Parametrization and computations in shape spaces with area and boundary invariants.

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***Abstract*— Shape spaces play an important role in several applications in robotics, most notably by providing a manifold structure on which to perform motion planning, control, behavior discovery and related algorithmic operations. Many classical approaches to defining shape spaces are not well suited to the needs of robotics. In this abstract, we outline an approach to defining shape spaces that address the needs of such problems, which often involve constraints on area/volume, perimeter/boundary, etc. Using the simple example of the space of constant-area and constant-perimeter triangles, which are represented as Riemannian manifolds, we demonstrate efficient solutions to problems involving continuous shape evolution, optimal sampling, etc.**

I. Introduction

This work is concerned with shape spaces and their applications in robotics. The term *shape space* has been used to refer to various different concepts in the literature. One common definition [1] is based on a normalization that eliminates size, orientation and other effects. While this is useful in recognition and analysis tasks, this is a very inconvenient definition in applications involving robot motion planning and control. Instead, we deal with the space of curves (or surfaces, in higher dimensions) without such artificial normalization operations. In addition, we are interested in constrained shapes, such as the space of curves defined by a constant length or enclosing a constant area.

A concrete problem that motivates our work is that of metamorphic robots. We are interested in decentralized control of the shape of a robot collective and we seek efficient algorithms for motion planning for such spatially extended systems. Typical applications include formation control [2] and locomotion, e.g., the geometric swimming problem [3], [4] and its robotic analog [5], [6]. There is an unfulfilled need for efficient ways to encode such tasks and for suitable decentralized control algorithms. We take the view that the task encoding problem can be solved in a lower-dimensional setting and then used to define efficient decentralized planning and control strategies for the true high dimensional system. In this setting, the large number of units in the collective is not a computational cost but, in fact, a desirable way to drive the approximation error down, see e.g., [7]. In this abstract, we address the problem of encoding the constrained shape evolution of simple polygonal shapes. We work out the concrete case of simple triangle shapes but the approach may be extended to other polygonal shapes as well.

Our goal is to pose the space of all triangles, or other polygonal shapes, as a manifold on which we can compute geodesics, generate sample points, etc. We provide a suitable parametrization of the manifold and outline some useful algorithms using this parametrization.

II. Parametrization and computations in the shape space of unit-area and unit-perimeter triangles

We begin with a simple parametrization based on Bookstein coordinates. The three vertices of a triangle may be represented as a vector of complex numbers, $A = (1, -1, u + iv)$. We treat the base as fixed (assumed to be of length 2 here, but other scaling factors will not affect our argument) and ask the question of how to move the free vertex in such a way that the desired constraints may be satisfied. The shape space is then defined in terms of the feasible set of such points.

Given two triangles represented in the above vector notation, there exists a simple Euclidean distance between them, $d(A, B) = \sqrt{1- \| A^*B \|^2}$. Such a distance minimizes the quantity $\| e^{i\phi}A - B \|$.

In truth, this distance is a good measure only locally. If a triangle is perturbed slightly through the movement of $u + iv$ then we may compute the distance between two triangles using the normed vector distance. For large deviations, the true distance will be non-Euclidean and we should deal with it in a Riemannian geometric sense. In order to obtain this description, let us define the coordinates of a normalized unit-area triangle and a slightly perturbed version as τ_1 and τ_2 respectively, where,

$$\tau_1 = \begin{bmatrix} (1-\frac{u}{3}) - i\frac{v}{3} \\ (-1-\frac{u}{3}) - i\frac{v}{3} \\ \frac{2}{3}u + i\frac{2}{3}v \end{bmatrix}, \quad \tau_2 = \begin{bmatrix} (1-\frac{u+du}{3}) - i\frac{v+dv}{3} \\ (-1-\frac{u+du}{3}) - i\frac{v+dv}{3} \\ \frac{2}{3}(u+du) + i\frac{2}{3}(v+dv) \end{bmatrix} \tag{1}$$

These coordinates are normalized by the area, i.e., $\sqrt{v}$ to yield unit-area triangles (introducing any other constant scaling factor would not affect the argument). Using this, the distance may be expressed as a function of (u, v, du, dv) and this function may be expanded as a Taylor series. This expansion provides a Riemannian metric in the following form, $ds^2 = E(u,v)du^2 + 2F(u,v)dudv + G(u,v)dv^2$. Beginning with the coordinates in equation 1, the coefficients in the Riemannian metric are $E(u,v) = \frac{2(3+u^2)}{3v(3+u^2+v^2)}, F(u,v) = \frac{u(-3-u^2+v^2)}{3v^2(3+u^2+v^2)}, G(u,v) = \frac{u^4-2u^2(-3+v^2)+(3+v^2)^2}{6v^3(3+u^2+v^2)}$.

This metric shows that the space is distinctly non-Euclidean. What this really means is that sampling and approximation (implicit operations in many popular planning algorithms) using the Euclidean distance will result in inaccurate results. One may get a better sense of this space by looking at the Gauss curvature function of this space, $\kappa = \frac{72v^3}{(3+u^2+v^2)^3}$. This is visualized in figure 1.

With such a characterization of the shape manifold, we are in a position to perform the first computation of interest to us - geodesics that provide paths for continuous shape

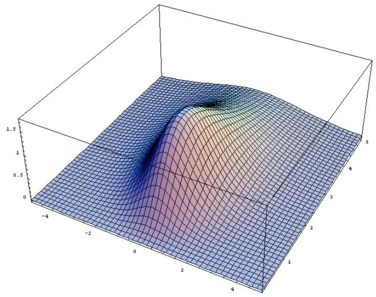

Fig. 1. Gauss curvature function for the space of unit area triangles.

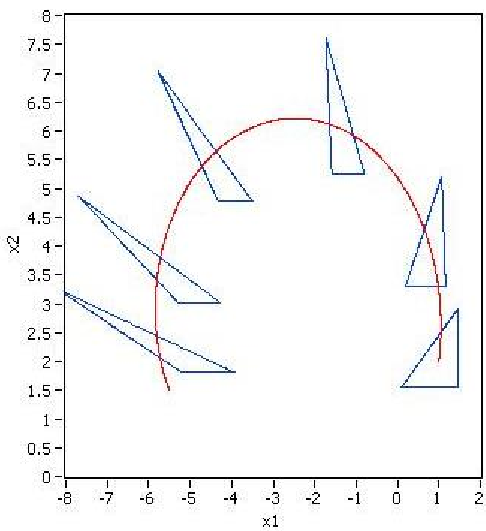

Fig. 2. A typical geodesic, indicating a path for continuous shape evolution, in the space of unit-area triangles.

evolution. The continuity these paths is very important in our application domain as large jumps will be difficult to handle in decentralized control algorithms that are implemented using largely local information.

The geodesics are computed using differential equations that have the form [8],

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0 \tag{2}$$

where Γ^i_{jk} are the Christoffel symbols and repeated indices are to be summed. Also, all coordinates are indicated by the generic x^i.

Figure 2 depicts typical geodesics along with the corresponding triangle shapes overlaid on the paths.

In addition to computing optimal paths and performing control, using techniques such as described in [9], we may also sample this space in a discovery or learning application. In [10], we presented techniques for the construction of low-discrepancy curves that optimally cover this space and can be used for sampling purposes. At a high level, these algorithms take the form,

1) Define a Riemannian metric and manifold structure for the space to be covered.

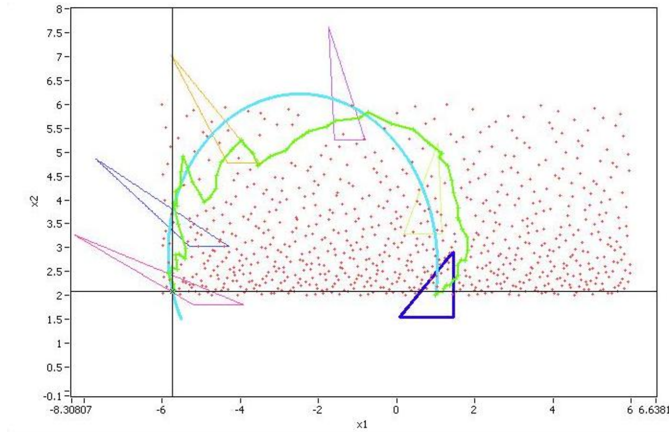

Fig. 3. Approximate geodesics computed on the basis of shortest path computations using optimally distributed landmark points in the shape space. Depicted on are low-discrepancy sample points distributed according to a Riemannian metric, the true geodesic (in blue) and the approximate geodesic (in green). Also, some points on the true geodesic include the triangle shape overlaid on the plot.

2) Define a diffeomorphism between a unit rectangle and this space.
3) Compute the diffeomorphism subject to constraints on fairness.

The input required by such algorithms is a parametrization of the shape manifold, which is exactly what we have provided above. A benefit of defining a Riemannian manifold that is covered in this way is the fact that we may define optimal low-discrepancy sample points in the shape space which reduces online geodesic computations, when a sub-optimal result is acceptable, to a shortest path problem in a directed graph. Such an approximate geodesic is depicted in figure 3.

Also, just as with area, we may also constrain the perimeter or boundary. These constraints often have physical meaning, e.g., constant area may refer to a nonpermeable cell membrane in a swimming microorganism while constant perimeter refers to a cell membrane that cannot be stretched or shrunk. Traditional approaches to defining shape spaces do not take such constraints into account in the computations, whereas in our formulation we are able to accommodate this.

We can carry out the above computations in an identical manner for the case of constant perimeter when we normalize equation 1 by the perimeter, $2 + \sqrt{(u-1)^2 + v^2} + \sqrt{(u+1)^2 + v^2}$. This yields a different Riemannian metric and the entire sequence of computation may be repeated.

III. Conclusions

We have presented a systematic procedure for defining constrained shape spaces and computing on such spaces. A concrete example has been worked out, for the case of triangles. The primary benefits of this methodology are twofold. Firstly, such a description provides the correct way to perform planning and control on these spaces. Secondly, the descriptions enable the application of some powerful search and space coverage algorithms that we have been developing in our research work. This has useful implications for applications requiring controlled shape evolution.

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